1.6 – 1.7

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Tensors

1. Represent physical quantities that are invariant but when given an explicit coordinate representation will typically have components that transform.
2. Live in flat tangent or cotangent spaces (or tensor products of these) at each point.

In terms of a labelling scheme (tangent, cotangent) we have seen:

$\underbrace{(0,0)}_{\text{scalar}}$ $\underbrace{(1,0)}_{\text{vectors}}$ $\underbrace{(0,1)}_{\text{dual vectors}}$ $\underbrace{(0,2)}_{\text{the metric}}$ $\underbrace{(2,0)}_{\text{the inverse metric}}$

$\eta_{\mu\nu}$ $\eta^{\mu\nu}$

In general you could have $(\overset{\text{upper}}{m}, \underset{\text{lower}}{n})$ mixed indices, e.g. $\underbrace{T^{\mu\nu\alpha}{}_{\lambda\beta\gamma\delta}}_{(3,4)\text{ tensor}}$

Though we will usually only deal with components, remember the full story is:

$$T = T^{\mu\nu\alpha}{}_{\lambda\beta\gamma\delta}\, \hat{e}_{(\mu)} \otimes \hat{e}_{(\nu)} \otimes \hat{e}_{(\alpha)} \otimes \hat{\omega}^{(\lambda)} \otimes \hat{\omega}^{(\beta)} \otimes \hat{\omega}^{(\gamma)} \otimes \hat{\omega}^{(\delta)}$$

A popular definition: A tensor is something that transforms like a tensor.

A better definition: Tensors are multi-linear maps from the space of vectors and dual vectors into the reals, i.e. $T(v,v,v,\omega,\omega,\omega,\omega) \to \#$

The latter definition is born out by a "well-fed" tensor:

Well-fed : $T^{\mu\nu\alpha}{}_{\lambda\beta\gamma\delta} V^\lambda V^\beta V^\gamma V^\delta \omega_\mu \omega_\nu \omega_\alpha \in \mathbb{R}$ no "free" indices

contrast with

Starving : $T^{\mu\nu\alpha}{}_{\lambda\beta\gamma\delta} V^\lambda V^\beta V^\gamma \omega_\mu \omega_\nu = F^\alpha{}_\delta$

Over-fed: $T^{\mu\nu\alpha}{}_{\lambda\beta\gamma\delta} V^\lambda V^\beta V^\gamma V^\delta V^\eta \omega_\mu \omega_\nu \omega_\alpha = H^\eta$

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The work we did on vectors and dual-vectors pays off big time now!

The transformation of tensor components is determined by their index structure:

$c \rightarrow c' = c$

$V^\mu \rightarrow V^{\mu'} = \Lambda^{\mu'}{}_{\mu} V^\mu$

$W_\mu \rightarrow W_{\mu'} = \Lambda^{\mu}{}_{\mu'} W_\mu$

$T^{\mu\nu} \rightarrow T^{\mu'\nu'} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} T^{\mu\nu}$

$H^\mu{}_\nu \rightarrow H^{\mu'}{}_{\nu'} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu}{}_{\nu'} H^\mu{}_\nu$

$G_{\mu\nu} \rightarrow G_{\mu'\nu'} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} G_{\mu\nu}$

etc.

Remember if you want to use matrix multiplication you need to get repeated indices adjacent

using $\Lambda_\nu{}^{\nu'} = (\Lambda^{\nu'}{}_\nu)^T$

and $\Lambda^\nu{}_{\nu'} = (\Lambda^{\nu'}{}_\nu)^{-1}$

thus $\Lambda_{\nu'}{}^{\nu} = (\Lambda^{\nu'}{}_\nu)^{-1\,T}$

$\Lambda \equiv \Lambda^{\mu'}{}_\mu$ or $\Lambda^{\nu'}{}_\nu$

$$\begin{cases} \Lambda T \Lambda^T = T' \\ \Lambda H \Lambda^{-1} = H' \\ \Lambda^{-1\,T} G \Lambda^{-1} = G' \end{cases}$$

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Now that we have many tensors to consider we should be careful with some special ones.

$\eta_{\mu\nu}$ - takes an upper (tangent space) index to a lower one, i.e. $\eta_{\mu\nu} T^{\nu} = T_{\mu}$
$\eta^{\mu\nu}$ - vice-versa

both are symmetric, $\eta^{\mu\nu} = \eta^{\nu\mu}$, $\eta_{\mu\nu} = \eta_{\nu\mu}$

A very important distinction is the following:

$$\eta_{\mu\nu} T^{\nu} = T_{\mu} \qquad M_{\mu\nu} T^{\nu} = H_{\mu}$$

In both cases the upper ν index dropped to a μ, but only for the action of $\eta_{\mu\nu}$ does the result correspond to the dual of what you start with. For a general tensor like $M_{\mu\nu}$ above result is a different quantity.

We can consider some η on η action! $\eta^{\alpha\mu}\eta^{\beta\nu}\eta_{\mu\nu} = \eta^{\alpha\beta}$

$$(\eta^{\alpha\mu}\eta_{\mu\nu})\eta^{\beta\nu} = \eta^{\alpha\beta}$$

$$\delta^{\alpha}{}_{\nu}\,\eta^{\beta\nu} = \eta^{\alpha\beta}$$

$$\eta^{\beta\alpha} = \eta^{\alpha\beta} \checkmark$$

What about $\eta^{\mu\nu}\eta_{\mu\nu}$? $\eta^{\mu\nu}\eta_{\mu\nu} = \delta^{\nu}{}_{\nu} = \delta^{0}{}_{0} + \delta^{1}{}_{1} + \delta^{2}{}_{2} + \delta^{3}{}_{3} = 4$

We know the coordinates x^{μ} transform like vector components V^{μ}.
What about $\frac{\partial}{\partial x^{\mu}}$? Well $\frac{\partial}{\partial x^{\mu}}(x^{\nu}) = \delta_{\mu}{}^{\nu}$ = number so $\frac{\partial}{\partial x^{\mu}}$ must transform oppositely of x^{μ}, i.e. $\frac{\partial}{\partial x^{\mu}} = \partial_{\mu}$ is a dual vector.

Tensor Equations

The physical world has no pre-determined absolute coordinate system (inertial or otherwise). Truly fundamental laws of physics should be built out of quantities which are at least invariant under transformations connecting inertial frames (relativity). This means they should be built out of tensors!

But if we are working only with tensor components, how do we see that an equation is invariant?

We could use only scalar equations: $A = B \Rightarrow \underset{A}{\overset{\shortparallel}{A'}} = \underset{B}{\overset{\shortparallel}{B'}}$

But we want more interesting options (like $\vec{F} = m\vec{a}$). For more general scenarios, when working w/ components we just need to make sure that the equation is covariant, i.e. each side transforms the same way!

$$A^{\mu\nu} = B^{\mu} C^{\nu}{}_{\alpha} D^{\alpha} \Rightarrow A^{\mu'\nu'} = B^{\mu'} C^{\nu'}{}_{\alpha'} D^{\alpha'}$$

$$\Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} A^{\mu\nu} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} B^{\mu} C^{\nu}{}_{\alpha} \overbrace{\Lambda^{\alpha}{}_{\alpha'} \Lambda^{\alpha'}{}_{\alpha}}^{\mathbb{I}} D^{\alpha}$$

$$\Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} (\quad) = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} (\quad)$$

Then when we restore the complete expressions w/ basis vectors and basis dual vectors, the complete expression will actually be invariant!

Index symmetries should also agree: $\underbrace{F_{\mu\nu}}_{F_{\mu\nu} = -F_{\nu\mu}} = \underbrace{\partial_\mu A_\nu - \partial_\nu A_\mu}_{\mu\nu = -\nu\mu}$

1.8

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To get some appreciation for the power and importance of tensors, consider Maxwell's equations:

$$\begin{array}{ll}
\vec{\nabla}\times\vec{B} - \partial_t \vec{E} = \vec{J} & 3 \\
\vec{\nabla}\cdot\vec{E} = \rho & 1 \\
\vec{\nabla}\times\vec{E} + \partial_t \vec{B} = 0 & 3 \\
\vec{\nabla}\cdot\vec{B} = 0 & \underline{1} \\
 & 8 \text{ equations}
\end{array}$$

While it is obvious that these equations are invariant under rotations, it is not obvious that they are invariant under boosts.

Introduce: $J^\mu = \begin{pmatrix} \rho \\ J^1 \\ J^2 \\ J^3 \end{pmatrix}$ and the $(0,2)$ field strength $F_{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix}$

Note: $F_{\mu\nu} = -F_{\mu\nu}$

Then:

$$\begin{array}{lll}
\partial_\mu F^{\mu\nu} = J^\nu & 4 & (0,1,2,3) \\
\partial_{[\mu} F_{\nu\lambda]} = 0 & \underline{4} & (012, 013, 023, 123) \\
 & 8 \text{ eqns.} &
\end{array}$$

Now that we have expressed Maxwell's equations in terms of tensors (components) then we know that they are invariant under all Λ's including boosts!
But recall that these predict c, therefore c is invariant under boosts!

Relativistic Kinematics and Dynamics

3D Kinematics : $x^i(t) \to v^i(t) = \frac{dx^i}{dt} \to a^i(t) = \frac{dv^i}{dt}$

3D Dynamics : $\sum F^i(t) = ma^i = \frac{dp^i}{dt} \quad p^i \equiv mv^i$

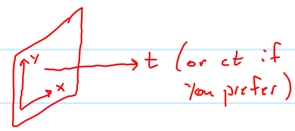
$\Big\}\ i = 1,2,3$

We know that in 4D SR $x^i \to x^\mu$ $\mu = 0,1,2,3$ and we expect $v^i \to u^\mu$.

2 issues with this:

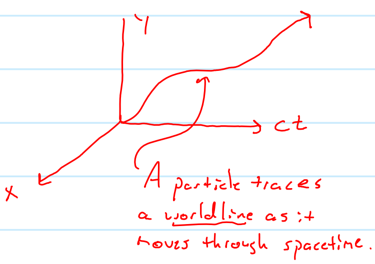
a) What is u^0?

b) If we try $u^\mu = \frac{\overbrace{dx^\mu}^{\text{vector}}}{\underbrace{dt}_{\text{component of vector}}} \Rightarrow$ overall not a vector !!



The problem originates in that for $x^i(t)$ we have a universal, invariant, monotonically increasing parameter t which can parameterize motion.

In SR, t is not universal or invariant So what can we use?



The length of the worldline is universal, invariant and monotonically increasing quantity.

Could we use $s = \int \sqrt{ds^2}$?

think about at rest

But: $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 < 0$ for motion w/ $v < c$.

So we use: $\tau = \int \sqrt{|ds^2|}$ as our parameter.

If the object is at rest : $\underbrace{ds^2 = -c^2dt^2}_{<0} \Rightarrow \tau = c\int dt$ ← "rest time"

So now: $x^i(t) \to x^\mu(\tau)$

$v^i = \frac{dx^i}{dt} \to u^\mu = \frac{\overbrace{dx^\mu}^{\text{vector}}}{\underbrace{d\tau}_{\text{scalar}}}$ = vector!

Lingo: $ds^2 < 0$ timelike (real objects)

$= 0$ lightlike (light)

> 0 spacelike (tachyonic?)

can be extended to all vectors

From this point forward we will do the sensible thing and set $c = 1$.

Doing so means we will use units of light-seconds for distance so $c = 1\ \frac{\text{light s}}{\text{s}}$.

Whenever we need/want to restore factors of c, we just put them in to make the units consistent, i.e. $\gamma = \frac{1}{\sqrt{1-v^2}} \to \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

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$U^\mu = \frac{dx^\mu}{d\tau}$ weirdness:

$$
\begin{aligned}
U^\mu U_\mu &= \eta_{\mu\nu} U^\mu U^\nu \\
&= \eta_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \\
&= \frac{\eta_{\mu\nu} dx^\mu dx^\nu}{d\tau^2} \\
&= \frac{\eta_{\mu\nu} dx^\mu dx^\nu}{-\eta_{\mu\nu} dx^\mu dx^\nu} \\
&= -1
\end{aligned}
$$

WTF?!

ell hat's unny

This might seem strange, usually $V_i V^i = V^2$ for $x^i(t)$.

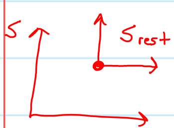
But consider: "$\vec{V}$" $= \left\langle \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \right\rangle$ ds = path length

"$\vec{V}\cdot\vec{V}$" $= \frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} = \frac{ds^2}{ds^2} = 1$

So this is just a consequence of how we are parameterizing paths.

Components of U^μ: $U^0(\tau) = \frac{dx^0}{d\tau} = \underbrace{\frac{dt}{d\tau}}_{c=1} = \frac{\gamma\, dt_{rest}}{d\tau} = \gamma \frac{d\tau}{d\tau} = \gamma$

in S w/ (t, x, y, z) $U^i(\tau) = \frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} = v^i \gamma$



$\gamma \equiv \frac{1}{\sqrt{1-v^2}}$

velocity connecting S to S_{rest}

$$\Downarrow$$

$$\boxed{U^\mu = \begin{pmatrix} \gamma \\ \gamma \vec{v} \end{pmatrix}}$$

$\hookrightarrow U_\mu U^\mu = -\gamma^2 + \gamma^2 v^2 = -\frac{1}{1-v^2} + \frac{v^2}{1-v^2} = -1$

In S_{rest}: $U^\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (again $U_\mu U^\mu = -1$)

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For momentum: $p^i \rightarrow p^\mu \equiv \underbrace{m}_{\text{mass}} U^\mu = \begin{pmatrix} m\gamma \\ m\gamma\vec{v} \end{pmatrix}$

If $v^2 \ll \overset{c^2}{1} \Rightarrow \gamma \approx 1 + \frac{1}{2}v^2 + \dots \Rightarrow \begin{cases} m\gamma \simeq \overset{\text{rest energy}}{m} + \overbrace{\tfrac{1}{2}mv^2}^{\text{non-relativistic KE}} + \dots = E \\ m\gamma v^i \simeq \underbrace{mv^i}_{\text{non-relativistic momentum}} + \mathcal{O}(v^{i3}) + \dots \equiv p^i \end{cases}$

relativistic energy and momentum!

So: $P^\mu = mu^\mu = \begin{pmatrix} E \\ \vec{p} \end{pmatrix}$

Then: $p_\mu p^\mu = m^2 U_\mu U^\mu = \underbrace{-m^2 = -E^2 + p^2}_{E^2 = p^2 + m^2}$ (w/ c $E^2 = p^2c^2 + m^2c^4$)

Even though our parameterization doesn't work when $m^2 = 0$, this result does!

$$p_\mu p^\mu = \begin{cases} <0 & m^2 > 0 & ds^2 < 0 & \text{timelike} \\ =0 & m^2 = 0 & ds^2 = 0 & \text{lightlike} \\ >0 & m^2 < 0 & ds^2 > 0 & \text{spacelike (tachyonic)} \end{cases}$$

We could try to relativize F^i to get $\sum F^\mu = \frac{dp^\mu}{d\tau}$ which will combine the work-energy and impulse-momentum theorems. But our primary concern is the gravitational force which will play out a bit differently.